# NEW TRANSIENCE BOUNDS FOR LONG WALKS IN WEIGHTED DIGRAPHS

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Fix two nodes $i$ and $j$ in an edge-weighted diagraph and form the following sequence: Let $a(n)$ be the maximum weight of walks from $i$ to $j$ of length $n$; if no such walk exists, $a(n) = -\infty$. It is known that, if $G$ is strongly connected, the sequence $a(n)$ is always eventually periodic with linear defect, i.e., after the *transient*, $a(n+p) = a(n) + p \cdot \lambda$. In fact, the ratio $\lambda$ is the largest mean weight of cycles in $G$. We call these cycles *critical*. Periodicity stems from the fact that the weights of critical cycles eventually dominate the maximum weight walks.

In this paper, we show two new asymptotically tight upper bounds on transients in weighted digraphs, taking into account the graph parameters cyclicity and girth. The previously best bound of Hartmann and Arguelles (Math. Oper. Res. 24, 1999) is, in general, incomparable with both our bounds. The significant benefit of our two new bounds is that each of them turns out to be linear in the number of nodes in various classes of weighted digraphs, whereas Hartmann and Arguelles' bound is intrinsically at least quadratic. In particular, our bounds are linear for (bi-directional) trees.

We hence prove the following two upper bounds on the transient in strongly connected digraphs with $N$ nodes. They both contain the (unweighted) index of convergence $\mathrm{ind}(G)$, i.e., the transient of $G$ when choosing all weights to be equal. The term $\|G\|$ denotes the difference of the maximum and minimum weight of edges in $G$. The mean weight of critical cycles is denoted by $\lambda$, and $\lambda_{\mathrm{nc}}$ denotes the largest mean weight of cycles that have no node on critical cycles. Denote by $G_c$ the subgraph induced by the critical cycles.

**Theorem 1** (Repetitive Bound)**.** *Denoting by $\hat{g}$ the maximum girth of strongly connected components of $G_c$, the transient between any two nodes is at most*

$$\max\left\{\frac{\|G\|\cdot(3N-2+\mathrm{ind}(G))}{\lambda-\lambda_{\mathrm{nc}}}\ ,\ (\hat{g}-1)+2\,\hat{g}\cdot(N-1)\right\}\quad .$$

**Theorem 2** (Explorative Bound)**.** *Denoting by $\hat{\gamma}$ and $\widehat{\mathrm{ind}}$ the maximum cyclicity and maximum index of strongly connected components of $G_c$, respectively, the transient between any two nodes is at most*

$$\max\left\{\frac{\|G\|\cdot(3N-2+\mathrm{ind}(G))}{\lambda-\lambda_{\mathrm{nc}}}\ ,\ (\hat{\gamma}-1)+2\,\hat{\gamma}\cdot(N-1)+\widehat{\mathrm{ind}}\right\}\quad .$$

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